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Erratum to: Homology stability for outer automorphism groups of free groups

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Abstract We correct the proof of Theorem 5 of the paper *Homology stability for outer automorphism groups of free groups*, by the first two authors.

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In [2] a proof was presented that the homology of certain groups $\Gamma_{n,s}$ is independent of both n and s for n sufficiently large. The groups $\Gamma_{n,s}$ include $Aut(F_n)$ $(s = 1)$ and $Out(F_n)$ $(s = 0)$. In August of 2005 Nathalie Wahl discovered an error in the proof, and the purpose of this note is to fix that error.

We assume the reader is familiar with [2], whose notation and conventions we will use here without further comment. The error occurs in the first part of the proof of Theorem 5, showing that the map $\beta_*\colon H_i(\Gamma_{n,s+2}) \to H_i(\Gamma_{n+1,s})$ is injective for $n >> i$ and $s \geq 1$. The argument used a diagram chase in the following diagram:

$$\begin{array}{ccc}
H_i(\Gamma_{n,s+2}, \Gamma_{n-1,s+4}) & \longrightarrow & H_{i-1}(\Gamma_{n-1,s+4}) \\
{\scriptstyle \beta_*}\big\downarrow & & {\scriptstyle \beta_*}\big\downarrow \\
H_i(\Gamma_{n+1,s}, \Gamma_{n,s+2}) & \longrightarrow & H_{i-1}(\Gamma_{n,s+2})
\end{array}$$

It was asserted that the top horizontal and right vertical arrows were successive maps in the long exact sequence of the pair $(\Gamma_{n,s+2}, \Gamma_{n-1,s+4})$, and hence their composition was the zero map, but in fact the group $\Gamma_{n,s+2}$ in the lower right corner of the diagram is a different subgroup of $\Gamma_{n+1,s}$ from the $\Gamma_{n,s+2}$ in the upper left corner, so that β_* is not induced by the inclusion map of the pair. It is in fact true that the composition is the zero map for n sufficiently large, but a proof seems to require the results proved in this correction.

We correct the problem by giving a completely new proof of stability with respect to s for $s \geq 1$, complementing the earlier proof of stability with respect to n. The new proof entirely avoids the diagram displayed above and instead focuses on the map $\mu\colon \Gamma_{n,s} \to \Gamma_{n,s+1}$. In the range where α_* is an isomorphism, the relation $\alpha = \beta\mu^2$ derived in section 2 of [2] shows that β_* is an isomorphism if and only if μ_* is an isomorphism. Note that μ_* is always injective since it has a left inverse obtained by gluing a disk to one of the new

boundary components, so we only need to prove surjectivity of μ_* in a stable range. The commutative diagram

$$\begin{tikzcd}
H_i(\Gamma_{n,s}) \arrow[r, "\alpha_*"] \arrow[d, "\mu_*"] & H_i(\Gamma_{n+1,s}) \arrow[r, "\alpha_*"] \arrow[d, "\mu_*"] & \cdots \arrow[r, "\alpha_*"] & H_i(\Gamma_{n+k,s}) \arrow[d, "\mu_*"] \\
H_i(\Gamma_{n,s+1}) \arrow[r, "\alpha_*"] & H_i(\Gamma_{n+1,s+1}) \arrow[r, "\alpha_*"] & \cdots \arrow[r, "\alpha_*"] & H_i(\Gamma_{n+k,s+1})
\end{tikzcd}$$

shows that if μ_* is an isomorphism for $n >> i$ then it will be an isomorphism in the same range that α is an isomorphism, namely $n \geq 2i+2$. It is thus not necessary to keep track of the precise stable range in the arguments given in this correction.

For $s \geq 2$ let $\eta\colon \Gamma_{n,s} \to \Gamma_{n+1,s-1}$ be induced by gluing a copy of the 3-punctured sphere $M_{0,3}$ to the first and last boundary components of $M_{n,s}$. The stabilization α is the composition $\eta\mu\colon \Gamma_{n,s} \to \Gamma_{n+1,s}$, and now we want to consider the opposite composition $\mu\eta\colon \Gamma_{n,s} \to \Gamma_{n+1,s}$. We will show that $\mu\eta$ is an isomorphism on H_i for $n >> i$, and hence $\mu_*\colon H_i(\Gamma_{n,s-1}) \to H_i(\Gamma_{n,s})$ is surjective for $n >> i$ and $s \geq 2$. Since α is a homology isomorphism for $n >> i$, and, as we will see, α commutes with $\mu\eta$, it will suffice to show that $\mu\eta$ is a homology isomorphism after passing to the direct limit under stabilization by α. This turns out to be much easier than showing $\mu\eta$ is a homology isomorphism before passing to the limit.

To prove that $\mu\eta$ is a homology isomorphism in the limit we use a new simplicial complex $Z_{n,s}$, defined for $s \geq 2$. To simplify the notation we will omit the subscript s since it will be fixed throughout the proof, so we write $Z_{n,s}$ as Z_n and $M_{n,s}$ as M_n. A vertex of Z_n is an equivalence class of pairs (S,a) where S is a non-separating sphere in M_n and a is an embedded arc in M_n joining the first and last boundary spheres ∂_0 and ∂_{s-1} and intersecting S transversely in one point; we refer to a as a *dual arc* for S. The equivalence relation on such pairs (S,a) is given by isotopy of $S \cup a$ keeping the endpoints of a in ∂M_n. A set of $k+1$ vertices $(S_0,a_0),\cdots,(S_k,a_k)$ forms a k-simplex if $S_i \cup a_i$ is disjoint from $S_j \cup a_j$ for $i \neq j$ and the spheres S_i form a coconnected system (see Figure 1). This implies that no two S_i's or a_i's are isotopic. Note that arcs can always be made disjoint by general position, so the disjointness condition really only involves intersections between different spheres and between spheres and arcs.

We note that there is an equivalent way of viewing a simplex of Z_n in terms of enveloping sphere-pairs. For a simplex $\{(S_0,a_0),\cdots,(S_k,a_k)\}$, take two parallel copies of each S_i, one on either side of S_i, and join each of these new spheres to a sphere parallel to either ∂_0 or ∂_{s-1} by a tube following the half of a_i on the appropriate side of S_i. This produces a pair of spheres S_e, S'_e separating M_n into two components, one of which contains the spheres S_i. The only boundary spheres contained in this component are ∂_0 and ∂_{s-1}, and splitting this component along the spheres S_i produces two simply-connected pieces, one bounded by S_e, ∂_0, and the S_i's, the other bounded by S'_e, ∂_{s-1}, and the S_i's. Conversely, given a coconnected system $S_0,\cdots,S_k$ and two spheres S_e, S'_e with the properties just listed,

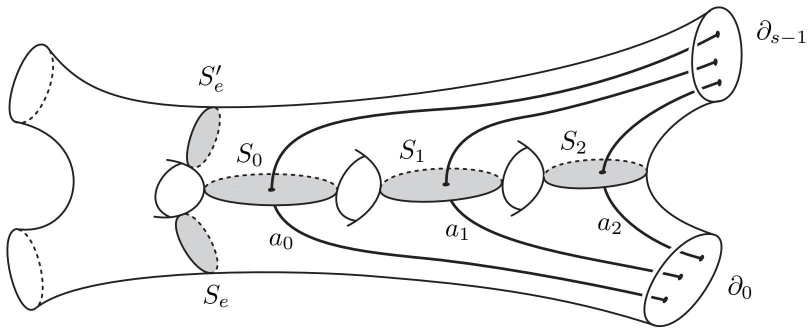

Figure 1: A 2-simplex in $Z_{n,s}$

then there are dual arcs a_i in the split-off submanifold such that $\{(S_0,a_0),\cdots,(S_k,a_k)\}$ is a simplex of Z_n, and these a_i's are unique up to isotopy.

We wish to describe now an inclusion $Z_n \hookrightarrow Z_{n+1}$. This will be induced by an inclusion $M_n \hookrightarrow M_{n+1}$. We have already used one such inclusion in the definition of the stabilization $\alpha\colon \Gamma_{n,s} \to \Gamma_{n+1,s}$ when we regarded M_{n+1} as being obtained from M_n by attaching $M_{1,2}$ to ∂_0 along one boundary sphere of $M_{1,2}$. However, an alternative approach will make things a little clearer when dealing with the complexes Z_n. Here we build M_{n+1} from M_n by attaching $M_{1,1}$, identifying a disk in $\partial M_{1,1}$ with a disk in ∂_0. The first inclusion $M_n \hookrightarrow M_{n+1}$ is then recovered by attaching a product $S^2 \times I$ to the new ∂_0. Since attaching this product does not affect isotopy classes of diffeomorphisms modulo Dehn twists, the new inclusion $M_n \hookrightarrow M_{n+1}$ gives the same α as the old one.

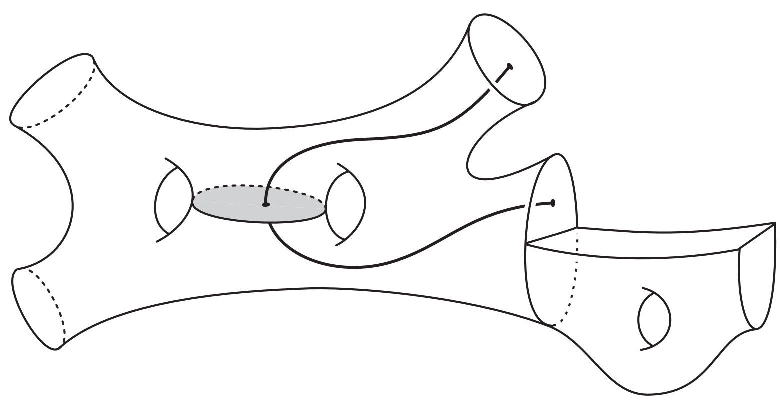

Figure 2: Stabilization by α and the inclusion $Z_n \to Z_{n+1}$

The new inclusion $M_n \hookrightarrow M_{n+1}$ induces a map $Z_n \to Z_{n+1}$ since we may assume simplices of Z_n are represented using pairs (S_i,a_i) whose arcs a_i are disjoint from the disk in ∂_0 where $M_{1,1}$ is attached. This map $Z_n \to Z_{n+1}$ is injective by general properties of sphere

systems (uniqueness of normal forms [1]). Let Z_∞ denote the direct limit of the complexes Z_n under these inclusions $Z_n \hookrightarrow Z_{n+1}$.

Lemma *Z_∞ is contractible.*

Proof Given a map $g\colon S^k \to Z_\infty$, we wish to extend this to a map $D^{k+1} \to Z_\infty$. We may assume g is simplicial with respect to some triangulation of S^k. This triangulation has finitely many simplices, so the image of g lies in Z_n for some n. Let $M_n \subset M_{n+1} \subset \cdots \subset M_{n+k+1}$ be the alternate inclusions described above inducing α, and choose a nonseparating sphere T_i in each $M_{n+i+1} - M_{n+i}$.

Triangulate D^{k+1} by coning off the triangulation of S^k to the centerpoint of D^{k+1}, a new vertex v. Define $g(v)$ to be (T_0, b_v) where b_v is any arc in M_{n+1} dual to T_0. Next, we extend g over each interior edge e of D^{k+1} in the following way. The endpoints of e map to (T_0, b_v) and to another vertex (S_0, a_0). Let g send the midpoint of e to (T_1, b_e), where b_e is an arc in M_{n+2} which is in the complement of the coconnected system $\{S_0, T_0\}$. Then $\{(S_0, a_0), (T_1, b_e)\}$ and $\{(T_1, b_e), (T_0, b_v)\}$ are edges of Z_{n+2} so g extends over e by mapping its two halves to these two edges.

The extension of g over simplices of D^{k+1} of higher dimension proceeds in a similar fashion, by induction on the dimension of the simplices. Each i-simplex σ of D^{k+1} not contained in S^k is the cone to v of an $(i-1)$-simplex in S^k. The map g sends this $(i-1)$-simplex to a possibly degenerate simplex $\{(S_0, a_0), \cdots, (S_{i-1}, a_{i-1})\}$ in Z_n. The rest of the boundary of σ is sent by induction to a subcomplex with additional vertices (T_j, b_τ) for $0 \le j < i$, where τ ranges over the faces of σ not in S^k. We send the barycenter of σ to (T_i, b_σ) where b_σ is chosen in M_{n+i+1} and in the complement of the coconnected system $\{S_0, \cdots, S_{i-1}, T_0, \cdots, T_{i-1}\}$. We can then extend g over σ by coning off to its barycenter. This gives the induction step, and at the end of the induction we have extended g over D^{k+1}. Since g and k were arbitrary, this shows Z_∞ is contractible. □

The natural action of Γ_n on Z_n is transitive on simplices of each dimension, since splitting M_n along the $k+1$ spheres of a k-simplex produces the manifold M_{n-k-1} with $2k+2$ new punctures, each joined to ∂_0 or ∂_{s-1} by an arc, and any two such configurations are diffeomorphic. The action of Γ_n on Z_n is compatible with the stabilization α, in the sense that for each $g \in \Gamma_n$ the following diagram commutes:

$$\begin{array}{ccc} Z_n & \xrightarrow{g} & Z_n \\ \downarrow & & \downarrow \\ Z_{n+1} & \xrightarrow{\alpha(g)} & Z_{n+1} \end{array}$$

Thus the direct limit group Γ_∞ acts on Z_∞. This action is also transitive on k-simplices for each k.

For the action of Γ_∞ on Z_∞ the stabilizer of a simplex includes group elements that permute the vertices of the simplex, so to avoid this we consider $U_\infty = \Delta(Z_\infty)$, the complex whose k-simplices are all the simplicial maps from the standard k-simplex to Z_∞. Note that U_∞ is the direct limit of the complexes $U_n = \Delta(Z_n)$. The homology of U_∞ is trivial since Z_∞ has trivial homology. The action of Γ_∞ on Z_∞ induces an action on U_∞. The quotient U_∞/Γ_∞ is contractible since it is the direct limit of the quotients U_n/Γ_n and these quotients are combinatorially the same as the quotients W_n/Γ_n in the proof of Theorem 4 of [2], which were $(n-2)$-connected.

The stabilizer of a vertex for the action of Γ_n on U_n is a copy Γ'_{n-1} of Γ_{n-1} in Γ_n, and the inclusion of this stabilizer is the map $\mu\eta$. The map $\mu\eta$ is induced by gluing a four-punctured 3-sphere to M_n by attaching two of its boundary spheres to ∂_0 and ∂_{s-1}. As in the case of α, there is an alternative description of $\mu\eta$ as being induced by gluing a twice-punctured 3-sphere to M_n by attaching its boundary spheres to ∂_0 and ∂_{s-1} along disks (see Figure 3).

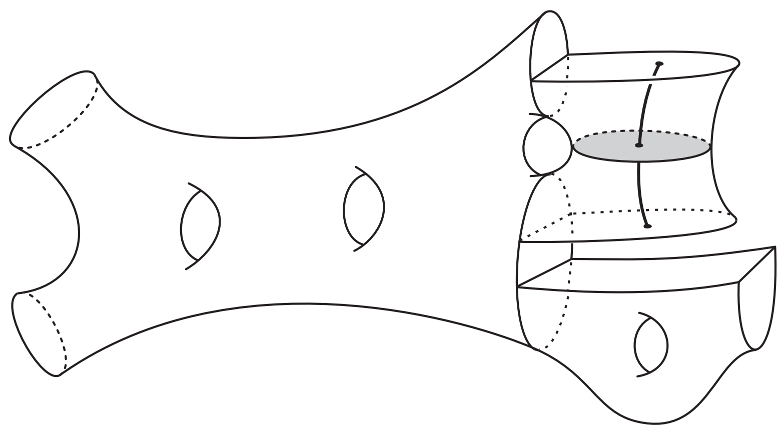

Figure 3: Compatibility of $\mu\eta$ with α

This alternative description of $\mu\eta$ makes it clear that $\mu\eta$ commutes with α, giving a commutative diagram:

$$\begin{array}{ccccccccc}
\Gamma'_{n-1} & \xrightarrow{\alpha} & \Gamma'_{n} & \xrightarrow{\alpha} & \Gamma'_{n+1} & \xrightarrow{\alpha} & \Gamma'_{n+2} & \xrightarrow{\alpha} & \dots \\
\downarrow{\scriptstyle\mu\eta} & & \downarrow{\scriptstyle\mu\eta} & & \downarrow{\scriptstyle\mu\eta} & & \downarrow{\scriptstyle\mu\eta} & & \dots \\
\Gamma_{n} & \xrightarrow{\alpha} & \Gamma_{n+1} & \xrightarrow{\alpha} & \Gamma_{n+2} & \xrightarrow{\alpha} & \Gamma_{n+3} & \xrightarrow{\alpha} & \dots
\end{array}$$

Thus in the limit action of Γ_∞ on U_∞ the inclusion of a vertex stabilizer is the direct limit map $\mu\eta\colon \Gamma'_\infty \to \Gamma_\infty$. Similarly, for stabilizers of higher dimensional simplices the inclusions of stabilizers are iterates of $\mu\eta$.

We can now use the equivariant homology spectral sequence arising from the action of Γ_∞ on U_∞ to prove:

Theorem *The map* $(\mu\eta)_*\colon H_i(\Gamma'_\infty) \to H_i(\Gamma_\infty)$ *is an isomorphism for each* $s \geq 2$.

As explained earlier, this implies:

Corollary *The map* $\mu_*\colon H_i(\Gamma_{n,s}) \to H_i(\Gamma_{n,s+1})$ *is an isomorphism when* $n \geq 2i+2$ *and* $s \geq 1$.

Proof of the theorem. The proof proceeds by induction on i. The equivariant homology spectral sequence has

$$E^1_{p,q} = \bigoplus_{\sigma_p} H_q(stab(\sigma_p)) \Rightarrow \widetilde{H}_{p+q}(U_\infty)$$

where $\{\sigma_p\}$ is a chosen set of orbit representatives for the p-simplices of U_∞. The differential $d^1\colon E^1_{0,i} \to E^1_{-1,i}$ is the map $\mu\eta\colon \Gamma'_\infty \to \Gamma_\infty$ we are interested in.

The j-th row of the E^1 page of the spectral sequence is a chain complex computing the homology of the quotient U_∞/Γ_∞ with local coefficients in the system of groups $H_j(stab(\sigma_p))$. Each face of the boundary of σ_p is equal to $h\sigma_{p-1}$ for some σ_{p-1} and some element $h \in \Gamma_\infty$, and the corresponding term of the d^1 map is the map $h_*\colon H_j(stab(\sigma_p)) \to H_j(stab(\sigma_{p-1}))$ induced by conjugation by h. For $j < i$ we may assume by induction that the vertical maps in the commutative diagram below are isomorphisms.

$$\begin{array}{ccc} H_j(stab(\sigma_p)) & \xrightarrow{h_*} & H_j(stab(\sigma_{p-1})) \\ \downarrow & & \downarrow \\ H_j(\Gamma_\infty) & \xrightarrow{h_*} & H_j(\Gamma_\infty) \end{array}$$

The lower h_* in this diagram is the identity since it is induced by an inner automorphism of Γ_∞. Thus the local coefficient system is trivial. (Here we are following a line of reasoning that can be found in Section 7.4 of [3].)

Since the quotient has trivial homology, this shows that the entire E^1 page below the i-th row is zero. The spectral sequence converges to 0 since U_∞ is contractible, and the only differential with a chance of killing $E^1_{-1,i}$ is $d^1 = \mu\eta$, proving that this map must be onto.

To finish the induction we need to show that $\mu\eta$ is in fact an isomorphism on $H_i(\Gamma'_\infty)$. Since $\mu\eta$ is surjective on $H_i(\Gamma'_\infty)$, it is also surjective as a map $H_i(\Gamma'_{n-1,s}) \to H_i(\Gamma_{n,s})$ for large n since α is an isomorphism on H_i for large n. Therefore $\mu_*\colon H_i(\Gamma_{n,s-1}) \to H_i(\Gamma_{n,s})$ is surjective for large n, and hence an isomorphism. Since $\alpha = \eta\mu$ this implies that $\eta_*\colon H_i(\Gamma'_{n,s}) \to H_i(\Gamma_{n+1,s-1})$ is an isomorphism for large n, so that $\mu\eta$ is also an isomorphism on $H_i(\Gamma'_{n,s})$ for large n, hence on $H_i(\Gamma'_\infty)$ as well. $\square$

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